Second order irreducible supersymmetry for periodic potentials

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Abstract. The technique of Darboux (supersymmetry) transformations is applied to periodic potentials. For the irreducible second order transformations the intermediate potentials can have poles while the final one is regular. A new kind of symmetry is detected, namely, translational invariance with respect to Darboux transformations. The necessary and sufficient conditions for a potential to have this invariance are formulated. New periodic and non-periodic exactly solvable potentials are found.

1. Introduction

A nice algebraic method of quantum mechanics leads to new classes of exactly solvable Schrödinger equations. The method is known under three different names: supersymmetric quantum mechanics, factorization method, and Darboux transformation. The first name became popular after Witten's paper [1] proposing a simple supersymmetric model in quantum field theory. The second method is due to Schrödinger [2]. The third name was born in the soliton theory and it is related to the Darboux paper published in 1882 [3]. The Darboux approach originated as well the general techniques of *intertwining operators*. The basic notions of the intertwining techniques are very simple and they were discovered and rediscovered by many authors. Probably the first deep investigation of this method was made by Delsart [4]

2. Darboux transformations and supersymmetry (short review)

Suppose we start with a certain initial Schrödinger equation

$$h_0\psi_E = E\psi_E, \quad h_0 = -\partial_x^2 + V_0(x), \quad x \in [a,b] \tag{1}$$

whose solutions are known for some values of the parameter E. This parameter, in general, may be complex but for simplicity we will suppose that it is real. To solve another Schrödinger equation

$$h_1\varphi_E = E\varphi_E, \quad h_1 = -\partial_x^2 + V_1(x), \quad x \in [a,b] \tag{2}$$

it is convenient to use an *intertwining operator* which will be denoted L. Its defining identity is

$$h_1 L = L h_0 \tag{3}$$

It is clear from (3) that if ψ_E is a solution of (1) then $\varphi_E = L\psi_E$ is a solution to (2).

It is well-known from the inverse scattering method that the intertwining operators L exist for wide families of Hamiltonians h_0 and h_1, but in general they are complicated and not easy to apply. To overcome this difficulty one imposes some restrictions on L. Below, we shall look for L in form of linear differential operators. If L is of the first order, one arrives at the well known formulae of Darboux

$$L = \partial_x - u'(x)/u(x), \quad V_1 = V_0 - 2(\ln u)'' \tag{4}$$

where u is an eigenfunction of the initial Hamiltonian h_0

$$h_0 u = \alpha u \tag{5}$$

In this case, the transformation operator and the new potential are completely defined by the function u, the reason why one calls u the *transformation function.* If one wants to assure that the transformation (4) does not introduce singularities, the sole condition imposed on u is that it should be nodeless solution to Eq.(5).

For the differential operators $A, B, \ldots$ it is convenient to use the formal (Laplace) conjugation operation $^+$, defined by the relations $\partial_x^+ = -\partial_x$, $(aAB)^+ = \overline{a}B^+A^+$, $a \in \mathbb{C}$. In particular, $L^+ = -\partial_x - u'(x)/u(x)$, $h_0^+ = h_0$ and $h_1^+ = h_1$. Applying the formal conjugation $^+$ to both sides of the intertwining relation (3) one obtains a similar relation for L^+

$$h_0 L^+ = L^+ h_1 \tag{6}$$

This means that L^+ leads from the solutions of the Schrödinger equation with the Hamiltonian h_1 to the solutions of a the same equation with the Hamiltonian h_0 i.e., in a sense it acts in an inverse direction than L (though it is not inverse to L since the product of L and L^+ is not the identity operator). Indeed, it is easy to show that L together with L^+ yield the following factorizations

$$L^+L = h_0 - \alpha, \quad LL^+ = h_1 - \alpha, \tag{7}$$

where α (present as well in (5)) is called the *factorization constant.*

Let us introduce now the matrix Hamiltonian (superhamiltonian)

$$\mathcal{H} = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}$$

and two nilpotent matrix operators (supercharges)

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \quad \mathcal{Q}^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix} \tag{8}$$

It is easy to see that the intertwining relations can be viewed as the commutation relations between the supercharges and the superhamiltonian. The factorized expressions (7) leads to an anticommuting relation for the supercharges

$$\left\{\mathcal{Q},\mathcal{Q}^{+}\right\} = \mathcal{H} - \alpha I, \quad [\mathcal{Q},\mathcal{H}] = 0, \quad [\mathcal{Q}^{+},\mathcal{H}] = 0 \tag{9}$$

This means that the operators $\mathcal{H}$, Q and Q^{+} close to a simple superalgebra, i.e., the algebraic structure underlying the traditional Darboux transformation is the supersymmetry.

Despite that the operators L and L^{+} have nontrivial kernels, $Lu = 0$, $L^{+}u^{-1} = 0$, they can be used to establish the mapping between the spaces of solutions of the initial and transformed Schrödinger's equations. Denote by T_{0E} the following 2-dimensional space $T_{0E} = \ker(h_0 - E)$ and by T_{1E} the space $T_{1E} = \ker(h_1 - E)$. When $E \neq \alpha$ the operator L maps T_{0E} onto T_{1E}, L^{+} yields the inverse mapping and the correspondence is one-to-one.

When $E = \alpha$ the space T_{0E} is spanned by u and $\tilde{u}$

$$\tilde{u} = u\int^{x} u^{-2}dx \in T_{0E} \tag{10}$$

and the space T_{1E} is spanned by $v = 1/u$ and $\tilde{v}$

$$\tilde{v} = u^{-1}\int^{x} u^{2}dx \in T_{1E} \tag{11}$$

It is easy to see that $L\tilde{u} = u^{-1} \in T_{1E}$ and $L^{+}\tilde{v} = u$. It follows that for $E = \alpha$ one can define a linear mapping $T_{0E} \longleftrightarrow T_{1E}$ by: $u \longleftrightarrow \tilde{v}$, $\tilde{u} \longleftrightarrow v$.

The above correspondence is very useful for finding all square integrable solutions of the transformed equation if the corresponding solutions for the initial equation are known. It is easy to show that when the transformation function u is nodeless and $E \neq \alpha$ the function $\varphi_E = L\psi_E$ is square integrable if and only if ψ_E is square integrable. To find all square integrable solutions of the transformed equation it is then sufficient to analyze the functions v and $\tilde{v}$. Note that three different cases are possible.

Case 1:

$$\alpha = E_0 \text{ and } u \in L^2(a,b) \implies v \notin L^2(a,b),\ \tilde{v} \notin L^2(a,b)$$

$$E_0 \in \mathrm{Sp}h_0, \quad E_0 \notin \mathrm{Sp}h_1, \quad E_k \in \mathrm{Sp}h_0 \text{ and } E_k \in \mathrm{Sp}h_1,\ k = 1,2,\ldots$$

The Hamiltonian h_0 has an additional discrete spectrum level with respect to h_1.

Case 2:

$$\alpha < E_0 \text{ and } v = u^{-1} \in L^2(a,b) \implies E = \alpha \notin \mathrm{Sp}h_0, \quad E = \alpha \in \mathrm{Sp}h_1 \tag{12}$$

$$E_k \in \mathrm{Sp}h_0 \text{ and } E_k \in \mathrm{Sp}h_1,\ k = 0,1,2,\ldots$$

The Hamiltonian h_1 has an additional discrete spectrum level with respect to h_0.
Case 3:

$$\alpha < E_0 \text{ and } v = u^{-1} \notin L^2(a,b),\ \tilde{v} \notin L^2(a,b) \implies \tag{13}$$
$$E_k \in \mathrm{Sp}h_0 \text{ and } E_k \in \mathrm{Sp}h_1,\ k = 0, 1, 2, \ldots$$

Both h_0 and h_1 have alike discrete spectrum i.e. they are strictly isospectral.

When h_1 and h_0 are isospectral one says that the supersymmetry is *broken* and when their spectra are different the supersymmetry is *exact*.

When the transformation operator is of higher order, we say that it generates a *higher order Darboux transformation*. It is possible to show that any such operator may be expressed as a chain of first order operators $L^{(N)} = L_N L_{N-1} \ldots L_1$, creating a chain of exactly solvable Hamiltonians $h_0 \to h_1 \to \ldots \to h_N$. The Hamiltonian h_N may be either isospectral with h_0 or their spectra may differ by a finite number of levels. Similarly to the first order case, the Nth order operator is defined with the help of N solutions u_k, $k = 1, \ldots, N$ of the initial equation

$$V_N = V_0 - 2[\ln W(u_1, \ldots, u_N)]'', \quad h_0 u_k = \alpha_k u_k \tag{14}$$

where $W(u_1, \ldots, u_N)$ is the Wronskian for the system of solutions $u_1, \ldots, u_N$. The Nth order operators $L^{(N)}$ and $(L^{(N)})^+$ factorize the polynomials build up of the Hamiltonians h_0 and h_N

$$(L^{(N)})^+ L^{(N)} = (h_0 - \alpha_1)(h_0 - \alpha_2) \ldots (h_0 - \alpha_N)$$
$$L^{(N)}(L^{(N)})^+ = (h_1 - \alpha_1)(h_1 - \alpha_2) \ldots (h_1 - \alpha_N)$$

the property which leads to the higher order superalgebra

$$\left\{Q, Q^+\right\} = (\mathcal{H} - \alpha_1 I)(\mathcal{H} - \alpha_2 I) \ldots (\mathcal{H} - \alpha_N I)$$

meaning that the underlying algebraic structure for a chain of transformations is the parasupersymmetry.

A very interesting feature of higher order transformations is that some intermediate Hamiltonians may be deprived of physical meaning while the final Hamiltonian is good from the physical point of view. In this case the Nth order transformation is called *irreducible* (*reducible* otherwise). There exist two kinds of irreducible transformations. The first type appears when there are complex intermediate factorization energies. In this case some transformation functions are complex-valued. This leads to intermediate complex potentials and the corresponding Hamiltonians cannot be defined as self-adjoint operators. The second case corresponds to the transformation functions which have zeros, and the intermediate Hamiltonians include singular potentials. In particular, a case when the transformation functions are adjacent discrete spectrum eigenfunctions was first considered by Krein [5]. The general case has recently been analyzed in [6].

The supersymmetric quantum models constructed with the help of such irreducible transformations have new interesting properties. In particular, it is possible to construct superhamiltonians with degenerate ground state (a characteristic of the broken supersymmetry) but some excited states are non-degenerate (which is inherent to exact supersymmetry). Hence, such models have properties of both exact and broken supersymmetry at once.

3. Periodic potentials

Let the initial potential be periodic $V_0(x+T) = V_0(x)$. It is well-known that the Schrödinger equation with a periodic potential always has *pseudoperiodic solutions* (see [7], p.31), called also *Bloch functions* [8]

$$\psi_E(x) = \beta\psi_E(x+T)\,, \quad \beta = \text{const} \in \mathbb{C} \tag{15}$$

The spectrum of the Schrödinger equation represents a sequence of allowed and forbidden bands. For forbidden E-values β is real and positive.

Let us denote the band edges by

$$E_{0'} < E_1 \leq E_{1'} < E_2 \leq E_{2'} < \ldots < E_j \leq E_{j'} < \ldots$$

The intervals $[E_{j\,'}, E_{j+1}]$, $j = 0, 1, \ldots$ belong to the spectrum of h_0. The functions $\psi_j(x)$ and $\psi_{j\,'}(x)$ have the same number of nodes j.

It is clear that to get a Darboux transformed potential which is periodic one has to use the Bloch functions. It is not difficult to show that when $\alpha \leq E_{0\,'}$ any Bloch function is nodeless and thus the transformed potential has exactly the same band structure as the initial one. We conclude that in this case the band structure is invariant with respect to Darboux transformations, i.e. the transformed and the initial potentials are always isospectral. It is not possible either to delete or create energy bands for a periodic potential with the help of the Darboux transformations generated by the Bloch functions and the corresponding supersymmetry is broken.

When the factorization constant is greater then $E_{0\,'}$ the transformation function has always nodes. Therefore the first order transformed potential is not regular in $\mathbb{R}$ (it has poles). Nevertheless, it is possible to show that when two factorization constants are taken in the same forbidden band, the corresponding Bloch functions lead to a nodeless Wronskian. This means that the second order transformed potential is regular on the whole real line. We thus get an irreducible second order Darboux transformation defining an irreducible second order supersymmetry.

4. New exactly solvable periodic potentials

To examine the consequencies, we have first of all applied our method to the Lamé potential

$$V(x) = n(n+1)\,m\,\text{sn}^2(x|m), \quad n \in \mathbb{N}, \quad 0 < m < 1 \tag{16}$$

For $n = 1$ the Bloch functions are specially simple to express for any E in terms of elliptic functions (see e.g. [9]). Fig. 1 shows the initial potential (a) together with the transformed one (b). Note a surprising result which could not be easily guessed from the formula (4). The transformed potential looks like a displaced copy of the initial one (at least up to our computer accuracy). For any non-singular first and second order Bloch families of Darboux transformations we have observed this phenomenon with different values of the displacement parameter. We will show further that this effect is exact. We call it *translational invariance with respect to Darboux transformation* [10]. A particular case of this invariance when the displacement is equal to half the period was already observed in [11, 12].

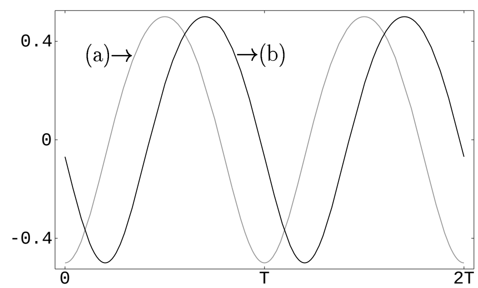

Fig. 1: Translational effect of the 2-nd order Darboux transformation. (a) The initial Lamé potential with $n = 1$ and $m = 0.5$; (b) The 2-susy equivalent. The factorization energies belong to the first energy gap $(E_1, E_{1'})$, the displacement $d = 0.747 \neq T/2$.

For other n the situation is different. For $n = 2$ our results are shown in the next figure below:

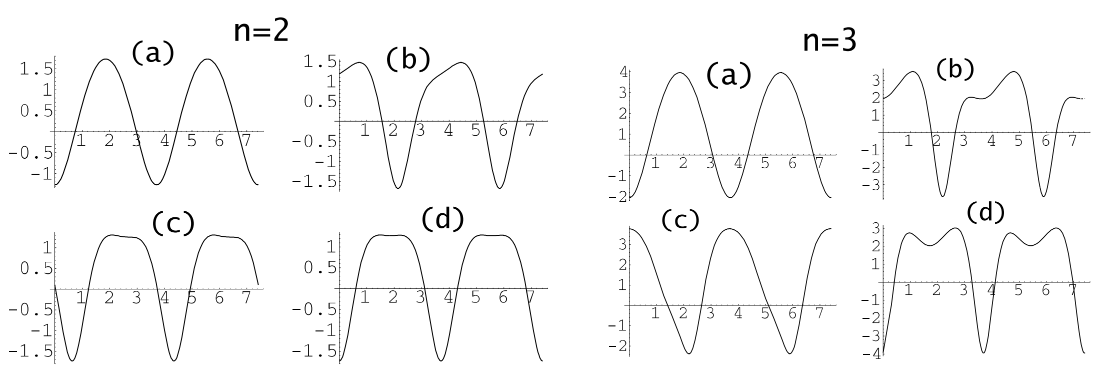

Fig. 2: A non-trivial result of the periodicity preserving second order Darboux transformations. (a) The original Lamé potentials, (b,c,d) the Darboux deformed versions.

Our following example is very similar to the well-known Kronig-Penney potential but it is easier to investigate since in contrast with the Kronig-Penney case our potential is continuous. We have considered the well-known soliton potentials restricted to a bounded subinterval of the real line and continued beyond by periodicity. An advantage of the so defined potentials is that both linearly independent solutions of the Schrödinger equation are given by elementary functions for any value of E. As a result one can get the condition for the band edges in form of a transcendental equation which may be easily solved numerically [13].

The Fig.3 shows the initial, periodically continued one-soliton potential together with its 2-nd order Darboux transformed versions. While the initial potential is not smooth at its maxima, the transformed ones have smooth maxima which are, however, displaced. The points of non differentiability, however, remain fixed at $\pm a + nT$. We see from these graphics that the transformed potentials tend to become displaced copies of the initial one but the "invariance" here is only approximate (see [10]).

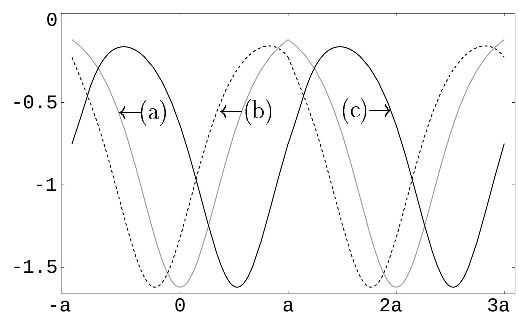

Fig. 3: Second order Darboux transformations of the periodically continued one soliton potential. (a) The initial potential; (b-c) The modified forms after using (14) with $u_1,\ u_2$ chosen to be the pairs of Bloch functions for $\alpha_1 = -10$ and $\alpha_2 = -2$.

The following Fig.4 shows our Darboux operation for a "piece-wise" two-soliton potential which by construction is smooth up to the first derivative at the maxima. The invariance here is again approximate but its features cannot be seen by the naked eye; the displacement looks as perfect as for the $n = 1$ Lamé potential.

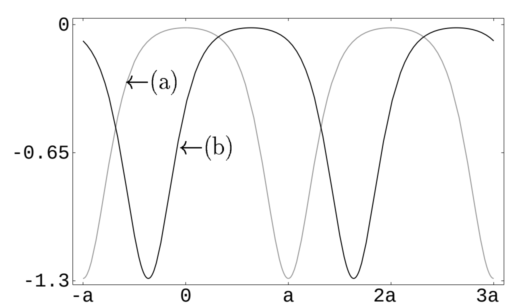

Fig. 4: Darboux operations on a periodically continued two soliton potential. (a) The initial potential; (b) the result of a 2-order Darboux transformation due to the pair of Bloch functions for $\alpha_1 = -0.6$, $\alpha_2 = -0.0007$. Notice a very good approximation to a finite displacement.

This example shows that the numerical experiments, while useful, can be misleading. To get a correct answer it is necessary to have an exact criterion whether a potential does or does not admit a supersymmetric displacement.

5. Translational invariance and potentials with periodicity defects

In fact, we can report the necessary and sufficient conditions for a potential to admit a Darboux displecement [14].

Theorem. *The Darboux transformation (4) can induce the finite displacement $V(x) \to V(x+d)$ if and only if the following quantity*

$$\alpha = \frac{1}{2}V(x) + \frac{1}{2}V(x+d) - \frac{1}{4}\left[\frac{V'(x)+V'(x+d)}{V(x)-V(x+d)}\right]^2 \tag{17}$$

is independent of x. Under this condition the potential can be as well displaced by $-d$. If this occurs, α is the factorization constant and the transformation function is defined by

$$[\ln u(x)]' = \pm\left[\frac{1}{2}V(x) + \frac{1}{2}V(x+d) - \alpha\right]^{1/2} \tag{18}$$

where $\pm$ signs correspond to d and to $-d$.

It is not difficult to check that the Lamé potential with $n = 1$ satisfies the equation (17) for continuous values of d. This justifies our conclusion obtained from numerical experiments in the precedent section.

Note that our last theorem is not restricted to the periodic potentials. It is possible to show that the one-soliton potential when defined on the entire real line satisfies this equation too. The two-soliton potential does not. This might mean that the translation invariance obtained for periodically continued soliton potentials *is not exact but it is only approximate*. In the case of one-soliton potential the condition (17) is violated at the points $x = \pm a + nT$.

Let us suppose now that the family of possible 1-st order displacements contains a triple d_1, d_2, d_3 such that $d_1 + d_2 + d_3 = 0$. Then the result of three corresponding Darboux transformations is an exact copy of the initial Hamiltonian. This means that the third order Darboux transformation operator $L(d) = L_3(d_3)L_2(d_2)L_1(d_1)$ is the symmetry operator for the initial Hamiltonian. It is known that the only potentials which admits a third order differential symmetry operator are expressed in terms of the Weierstrass elliptic functions

$$V(z) = 2\wp(z - z_0) + \mathrm{const}$$

This result together with our theorem means that when the domain of possible displacements is sufficiently large only the Weierstrass functions admit translation invariance with respect to Darboux transformation. The Lamé and one-soliton potentials are precisely the special cases of Weierstrass potential.

When the potential is the Weierstrass function, it is possible to express the factorization constant as a function of the displacement parameter $\alpha(d) = -\wp(d)$. Our necessary and sufficient condition becomes now the well known addition theorem for the Weierstrass function

$$\wp(u) + \wp(v) + \wp(u+v) = \frac{1}{4}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]$$

implying that any function satisfying this addition law is the Weierstrass function (or its degenerate form such as 1-soliton potential). This statement is a particular case of the known Weierstrass theorem (see e.g. [9]).

Starting from the expression (18) one can get the known formula for Bloch solutions of the Schrödinger equation with the Weierstrass potential

$$u(x) = \frac{\sigma(x + \omega' + d)}{\sigma(x + \omega')} e^{-x\zeta(d)} \tag{19}$$

Here sigma and zeta are the non-elliptic Weierstrass functions, ω' is the imaginary half-period of $\wp$.

It is interesting that all formulae are valid not only for the real d's but for a complex d of particular form: $d = d_0 + \omega'$ ($d_0 \in \mathbb{R}$). This permits one to get Bloch functions for all real values of the spectral parameter E and to obtain the known expressions for the band edges

$$E_{0'} = -\wp(\omega) = e_1, \quad E_1 = -\wp(\omega + \omega') = e_2, \quad E_{1'} = -\wp(\omega') = e_3$$

where ω is the real half-period of the Weierstrass function $\wp$. Note as well that by changing the sign of d in (19) one gets another linearly independent solution of the Schrödinger equation at the same value of the energy E, and hence, a general solution is now to our disposal.

Once we know the general solution of the Schrödinger equation with the Weierstrass potential for any E, we can use it in order to implement the Darboux transformations. In such a way we shall obtain new non-periodic potentials whose spectrum will consist of the band spectrum of the initial potential and a finite number of bound states encrusted in the forbidden energy bands. The Fig.5 shows some examples of those potentials which can describe periodicity defects inserted in a periodic lattice.

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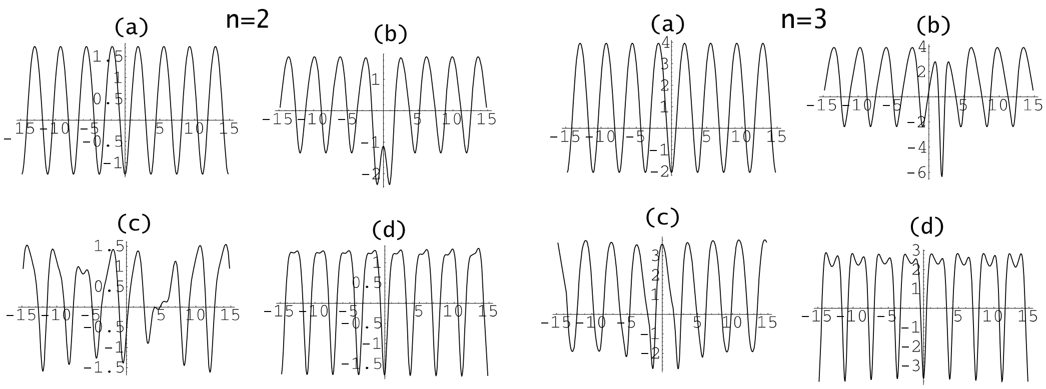

Fig. 5: Non-periodic deformations (b,c,d) of Lam'e (a) potentials

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