Complement E_{XIII}

EXERCISES

1. Consider a one-dimensional harmonic oscillator of mass m, angular frequency ω_0 and charge q. Let $| \varphi_n \rangle$ and $E_n = (n + 1/2)\hbar\omega_0$ be the eigenstates and eigenvalues of its Hamiltonian H_0.

For $t < 0$, the oscillator is in the ground state $| \varphi_0 \rangle$. At $t = 0$, it is subjected to an electric field "pulse" of duration τ. The corresponding perturbation can be written :

$$W(t) = \begin{cases} - q\mathscr{E}X & \text{for } 0 \leqslant t \leqslant \tau \\ 0 & \text{for } t < 0 \text{ and } t > \tau \end{cases}$$

$\mathscr{E}$ is the field amplitude and X is the position observable. Let $\mathscr{P}_{0n}$ be the probability of finding the oscillator in the state $| \varphi_n \rangle$ after the pulse.

a. Calculate $\mathscr{P}_{01}$ by using first-order time-dependent perturbation theory. How does $\mathscr{P}_{01}$ vary with τ, for fixed ω_0?

b. Show that, to obtain $\mathscr{P}_{02}$, the time-dependent perturbation theory calculation must be pursued at least to second order. Calculate $\mathscr{P}_{02}$ to this perturbation order.

c. Give the exact expressions for $\mathscr{P}_{01}$ and $\mathscr{P}_{02}$ in which the translation operator used in complement F_V appears explicitly. By making a limited power series expansion in $\mathscr{E}$ of these expressions, find the results of the preceding questions.

2. Consider two spin 1/2's, $\mathbf{S}_1$ and $\mathbf{S}_2$, coupled by an interaction of the form $a(t)\mathbf{S}_1 \cdot \mathbf{S}_2$; $a(t)$ is a function of time which approaches zero when $|t|$ approaches infinity, and takes on non-negligible values (on the order of a_0) only inside an interval, whose width is of the order of τ, about $t = 0$.

a. At $t = -\infty$, the system is in the state $| +, - \rangle$ (an eigenstate of S_{1z} and S_{2z} with the eigenvalues $+\hbar/2$ and $-\hbar/2$). Calculate, without approximations, the state of the system at $t = +\infty$. Show that the probability $\mathscr{P}(+ - \longrightarrow - +)$ of finding, at $t = +\infty$, the system in the state $| -, + \rangle$ depends only on the integral $\int_{-\infty}^{+\infty} a(t)\, \mathrm{d}t$.

b. Calculate $\mathscr{P}(+ - \longrightarrow - +)$ by using first-order time-dependent perturbation theory. Discuss the validity conditions for such an approximation by comparing the results obtained with those of the preceding question.

c. Now assume that the two spins are also interacting with a static magnetic field $\mathbf{B}_0$ parallel to Oz. The corresponding Zeeman Hamiltonian can be written :

$$H_0 = - B_0(\gamma_1 S_{1z} + \gamma_2 S_{2z})$$

where γ_1 and γ_2 are the gyromagnetic ratios of the two spins, assumed to be different.

Assume that $a(t) = a_0 \, e^{-t^2/\tau^2}$. Calculate $\mathscr{P}(+ \, - \longrightarrow - \, +)$ by first-order time-dependent perturbation theory. With fixed a_0 and τ, discuss the variation of $\mathscr{P}(+ \, - \longrightarrow - \, +)$ with respect to B_0.

3. Two-photon transitions between non-equidistant levels

Consider an atomic level of angular momentum $J = 1$, subject to static electric and magnetic fields, both parallel to Oz. It can be shown that three non-equidistant energy levels are then obtained. The eigenstates $| \varphi_M \rangle$ of J_z ($M = -1, 0, +1$), of energies E_M correspond to them. We set $E_1 - E_0 = \hbar\omega_0$, $E_0 - E_{-1} = \hbar\omega_0'$ ($\omega_0 \neq \omega_0'$).

The atom is also subjected to a radiofrequency field rotating at the angular frequency ω in the xOy plane. The corresponding perturbation $W(t)$ can be written:

$$W(t) = \frac{\omega_1}{2} (J_+ \, e^{-i\omega t} + J_- \, e^{i\omega t})$$

where ω_1 is a constant proportional to the amplitude of the rotating field.

a. We set (notation identical to that of chapter XIII):

$$| \psi(t) \rangle = \sum_{M=-1}^{+1} b_M(t) \, e^{-iE_M t/\hbar} \, | \varphi_M \rangle$$

Write the system of differential equations satisfied by the $b_M(t)$.

b. Assume that, at time $t = 0$, the system is in the state $| \varphi_{-1} \rangle$. Show that if we want to calculate $b_1(t)$ by time-dependent perturbation theory, the calculation must be pursued to second order. Calculate $b_1(t)$ to this perturbation order.

c. For fixed t, how does the probability $\mathscr{P}_{-1,+1}(t) = |b_1(t)|^2$ of finding the system in the state $| \varphi_1 \rangle$ at time t vary with respect to ω? Show that a resonance appears, not only for $\omega = \omega_0$ and $\omega = \omega_0'$, but also for $\omega = (\omega_0 + \omega_0')/2$. Give a particle interpretation of this resonance.

4. Returning to exercise 5 of complement H_{XI} and using its notation, assume that the field $\mathbf{B}_0$ is oscillating at angular frequency ω, and can be written $\mathbf{B}_0(t) = \mathbf{B}_0 \cos \omega t$. Assume that $b = 2a$ and that ω is not equal to any Bohr angular frequency of the system (non-resonant excitation).

Introduce the susceptibility tensor χ, of components $\chi_{ij}(\omega)$, defined by:

$$\langle M_i \rangle(t) = \sum_j \text{Re} \left[\chi_{ij}(\omega) \, B_{0j} \, e^{i\omega t} \right]$$

with $i, j = x, y, z$. Using a method analogous to the one in § 2 of complement A_{XIII}, calculate $\chi_{ij}(\omega)$. Setting $\omega = 0$, find the results of exercise 5 of complement H_{XI}.

If we sum the terms we get

$$\sum R = \frac{8C}{3} \tag{22-103}$$

Similarly

$P_{1/2} \to S_{1/2}$	
$m_j = 1/2 \to m_j = 1/2$	$\lvert\langle\sqrt{1/3}\, Y_{10}\rvert\, \boldsymbol{\varepsilon}\cdot\mathbf{r}\,\lvert Y_{00}\rangle\rvert^2 = 0$
$1/2 \to -1/2$	$\lvert\langle-\sqrt{2/3}\, Y_{11}\rvert\, \boldsymbol{\varepsilon}\cdot\mathbf{r}\,\lvert Y_{00}\rangle\rvert^2 = 2C/3$
$-1/2 \to 1/2$	$\lvert\langle\sqrt{2/3}\, Y_{1,-1}\rvert\, \boldsymbol{\varepsilon}\cdot\mathbf{r}\,\lvert Y_{00}\rangle\rvert^2 = 2C/3$
$-1/2 \to -1/2$	$\lvert\langle-\sqrt{1/3}\, Y_{10}\rvert\, \boldsymbol{\varepsilon}\cdot\mathbf{r}\,\lvert Y_{00}\rangle\rvert^2 = 0$

Again

$$\sum R = \frac{4C}{3} \tag{22-104}$$

Thus the ratio of the intensities is

$$\frac{R(P_{3/2} \to S_{1/2})}{R(P_{1/2} \to S_{1/2})} = \frac{8C/3}{4C/3} = 2 \tag{22-105}$$

The reason for *summing* over all the initial states is that when the atom is excited, all the p-levels are equally occupied, since their energy difference is so tiny compared to the $2p - 1s$ energy difference. We also sum over all the final states if we perform an experiment that does not discriminate between them, as is the case for a spectroscopic measurement. In our calculation of the $2p \to 1s$ transition rate, we *averaged* over the initial m-states. There we were concerned with the problem of asking: "If we have N atoms in the $2p$ states, how many will decay per second?" The averaging came about because of the fact that under most circumstances, when N atoms are excited, about $N/3$ go into each one of the $m = 1, 0, -1$ states. Here, the fact that there are more levels in the $P_{3/2}$ state than there are in the $P_{1/2}$ state is relevant. There will be altogether six levels, (four with $j = 3/2$ and two with $j = 1/2$) and there will be on the average $N/6$ atoms in each of the states. The fact that there are more atoms in the $j = 3/2$ subset of levels just means that more decay, and that therefore the intensity will be larger.

Problems

1. A hydrogen atom is placed in an electric field $\mathbf{E}(t)$ that is uniform and has the time dependence

$$\begin{aligned} \mathbf{E}(t) &= 0 & t < 0 \\ &= \mathbf{E}_0\, e^{-\gamma t} & t > 0 \end{aligned}$$

What is the probability that as $t \to \infty$, the hydrogen atom, if initially in the ground state, makes a transition to the $2p$ state?

2. Repeat the above calculation with the time dependence of the electric field given by

$$\mathbf{E}(t) = \mathbf{E}_0\, e^{-\alpha^2 t^2}$$

and with the condition that the hydrogen atom be in its ground state at $t = -\infty$. [*Hint.* As a first step, modify Eq. 22-9 appropriately.] Discuss your result when the time-variation of the electric field is extremely slow.

3. Consider a harmonic oscillator described by

$$H = \frac{1}{2m} p_x^2 + \tfrac{1}{2} m\omega^2(t)\, x^2$$

where

$$\omega(t) = \omega_0 + \delta\omega \cos ft$$

and $\delta\omega \ll \omega_0$.
Calculate the probability that a transition occurs from the ground state, as a function of time, given that the system is in the ground state at $t = 0$. Use perturbation theory. Use the fact that for $n \neq 0$,

$$\begin{aligned} \langle n \,|\, x^2 \,|\, 0 \rangle &= \hbar/2\sqrt{2}m\omega \quad \text{for} \quad n = 2 \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

Can you derive this formula using the material from Chapter 7?

4. Suppose a particle of rest mass M decays into two particles of rest mass m_1 and m_2, respectively. Use the relativistic relation between energy and momentum to compute the density of states ρ that appears in (22-57). [*Hint.* There is only one independent momentum, say $\mathbf{p}$, and what is needed is

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^6}\, \delta\left(E_{\text{initial}} - \sum_{\substack{\text{final}\\ \text{states}}} E\right)$$

5. Consider the above calculation when the decay is of the form

$$A \to B + C + D$$

with particles C and D massless.
[*Hint.* There are now two independent momenta.]

6. In this problem the *adiabatic theorem* is to be illustrated. The theorem states that if the Hamiltonian is changed very slowly from H_0 to H, then a system in a given eigenstate of H_0 goes over into the corresponding eigenstate of H,