3. Express, by means of scattering phase shifts, the first three coefficients of the expansion of the elastic scattering cross section $\frac{d\sigma}{d\Omega}$ in terms of Legendre polynomials.

4. Calculate the differential cross section for scattering in a repulsive field $U = \frac{A}{r^2}$ in the Born approximation. ~~Repeat the calculation for the case of classical mechanics~~. Find the limits of applicability of the formulae obtained.

5. Find the discrete levels for a particle in the attractive field $U(r) = -U_0 \exp(-r/a)$ for $l = 0$. Find the scattering phase shift δ_0 for this potential and discuss the relation between δ_0 and the discrete spectrum.

6. Show that for a Coulomb field there is a one-to-one correspondence between the poles of the scattering amplitudes and the levels of the discrete spectrum.

Hint. Use the relation

$$e^{2i\delta_l} = \frac{\Gamma\left(l+1+\frac{i}{k}\right)}{\Gamma\left(l+1-\frac{i}{k}\right)}.$$

7. Determine how the scattering phase shift changes for a small change in the scattering potential. Find the expression for the scattering phase shift in the case in which the potential can be considered as a perturbation.

8. Calculate the scattering phase shifts of slow particles in the field $V = \alpha/r^3$. The particles are slow enough for the condition $\mu\alpha k/\hbar^2 \ll 1$ to be satisfied.

9. Find the total cross section for the elastic scattering of fast particles by a perfectly rigid sphere of radius a ($\lambda\!\!\!^{-} \ll a$, where $\lambda\!\!\!^{-}$ is the de Broglie wavelength).

10. Find in the Born approximation the differential and total cross sections for scattering in the fields:

(a) $U(r) = g^2 \frac{e^{-\alpha r}}{r}$,

(b) $U(r) = U_0\, e^{-\alpha^2 r^2}$,

(c) $U(r) = U_0\, e^{-\alpha r}$.

11. Using the Born approximation, find the differential and total cross sections for the elastic scattering of fast electrons

(a) by a hydrogen atom, (b) by a helium atom.

It is interesting to note that for the Coulomb field the classical limit is approached for small v, whereas for potentials that have a finite range a, such as are discussed in Sec. 19, the classical limit is approached when $(a/\lambda)^{\frac{1}{2}} \gg 1$, that is, for large v. This is because the "size" $|ZZ'e^2/\mu v^2|$ of the Coulomb field increases more rapidly than $\lambda = \hbar/\mu v$ as v decreases.

Problems

1. Show that the coefficients of scattering by a one-dimensional square well potential (like Fig. 14 except that $V_0 < 0$) are given by Eqs. (17.5) if the sign of V_0 is changed there and in the expression for α. Discuss the dependence of transmission coefficient on E in this case.

2. Show that Eqs. (18.4) and (18.7) are valid for a general binary collision if γ is given by (18.5); make use of conservation of energy and mass.

3. Show that, when a particle of mass m_1 collides elastically with a particle of mass m_2 that is initially at rest, all the recoil (mass m_2) particles are scattered in the forward hemisphere in the laboratory coordinate system. If the angular distribution is spherically symmetrical in the center-of-mass system, what is it for m_2 in the laboratory system?

4. Express the scattering wave function (19.1) outside the scattering potential (but not necessarily in the asymptotic region) as the sum of a plane wave and an infinite series of spherical Hankel functions of the first kind [see Eqs. (15.12)]. From this expression and the discussion of Eqs. (15.13), show that the scattered wave is purely outgoing, even inside of the asymptotic region.

5. What must V_0a^2 be for a three-dimensional square well potential in order that the scattering cross section be zero at zero bombarding energy (Ramsauer-Townsend effect)? Find the leading term in the expression for the total cross section for small bombarding energy. Note that both the $l = 0$ and the $l = 1$ partial waves must be included.

6. State clearly the assumptions that go into the derivation of Eq. (19.31), and verify that it is a suitable approximation for the total cross section at low bombarding energies when the $l = 0$ wave is in resonance.

7. Make use of Eq. (19.31) and the result of Prob. 5, Chap. IV, to obtain an approximate expression for the total scattering cross section by a particular potential in terms of the bombarding energy E and the binding energy ϵ of a particle in that potential, when E and ϵ are small in comparison with V_0.

8. Compute and make a polar plot of the differential scattering cross section for a perfectly rigid sphere when $ka = \frac{1}{2}$, using the first three partial waves ($l = 0, 1, 2$). What is the total cross section in this case, and what is the approximate accuracy of this result when the three terms are used?

9. Find a general expression for the phase shift produced by a scattering potential $V(r) = A/r^2$, where $A > 0$. Is the total cross section finite? If not, does the divergence come from small or large scattering angles, and why? What modifications are necessary in the calculation if $A < 0$? Are any difficulties encountered in this latter case?

10. Protons of 200,000 electron-volts energy are scattered from aluminum. The directly back scattered intensity ($\theta = 180°$) is found to be 96 per cent of that computed from the Rutherford formula. Assume this to be due to a modification of the Coulomb potential that is of sufficiently short range so that only the phase shift for $l = 0$ is affected. Is this modification attractive or repulsive? Find the sign and magnitude of the change in the phase shift for $l = 0$ produced by the modification.

singlet scattering vanishes. Check your results by observing that since (in units of $\hbar$)

$$\tfrac{1}{2}\boldsymbol{\sigma}_P + \tfrac{1}{2}\boldsymbol{\sigma}_N = \mathbf{S}$$

one has

$$\begin{aligned}\boldsymbol{\sigma}_P \cdot \boldsymbol{\sigma}_N &= 2\mathbf{S}^2 - 3 \\ &= 1 \quad \text{when acting on triplet state} \\ & -3 \quad \text{when acting on singlet state}\end{aligned}$$

Note that the amplitude is independent of m_S so that m_S must be the same in the initial and final spin states. There are three states in the triplet, all contributing an equal amount to the cross section, and only one to the singlet cross section. (*Caution.* In calculating amplitudes such as

$$\frac{1}{\sqrt{2}}\,(\chi_\uparrow^{(P)}\chi_\downarrow^{(N)} - \chi_\downarrow^{(P)}\chi_\uparrow^{(N)})(A + B\boldsymbol{\sigma}_P \cdot \boldsymbol{\sigma}_N)\,\frac{1}{\sqrt{2}}\,(\chi_\uparrow^{(P)}\chi_\downarrow^{(N)} - \chi_\downarrow^{(P)}\chi_\uparrow^{(N)})$$

the amplitudes are added for the four terms before squaring. Can you explain why?)

9. It can be shown that the solution of the $l = 0$ Schrödinger equation for the potential

$$V(r) = -2\beta\lambda^2 \frac{e^{-\lambda r}}{(\beta\, e^{-\lambda r} + 1)^2}$$

which behaves asymptotically like e^{-ikr} is

$$F(k,r) = e^{-ikr} \cdot \frac{2k(\beta\, e^{-\lambda r} + 1) + i\lambda(\beta\, e^{-\lambda r} - 1)}{(\beta\, e^{-\lambda r} + 1)(2k - i\lambda)}$$

The solution that behaves asymptotically like e^{+ikr} is $F(-k,r)$. Thus the regular solution, which vanishes at $r = 0$, is

$$u(r) = [F(k,0)\, F(-k,r) - F(-k,0)\, F(k,r)]$$

Use this information to obtain the scattering amplitude $F(k) = [S(k) - 1]/2ik$. Discuss the solution for various limiting cases.

References

Scattering theory is discussed in all of the textbooks listed at the end of this volume. In addition there exist a number of advanced treatises that are devoted to this subject alone. Most accessible to students at this level is knows only a finite number of phase shifts over a limited range of energy, and this does not allow one to apply either of the theorems.

Further reading for Chapter 16

Goldberger and Watson (1964) has long been regarded as the authoritative reference on scattering theory, although it has now been superseded to some extent by Newton (1982). Both of them are research tomes. The beginning student may find the treatment by Rodberg and Thaler (1967) to be more accessible. Wu and Ohmura (1962) is intermediate between the textbook and research levels.

Problems

16.1 Derive Eq. (16.5) from momentum and energy conservation.

16.2 Calculate the $\ell = 0$ phase shift for the repulsive δ shell potential, $V(r) = c\,\delta(r - a)$. Determine the conditions under which it will be approximately equal to the phase shift of a hard sphere of the same radius a, and note the conditions under which it may significantly differ from the hard sphere phase shift even though c is very large.

16.3 Show that $\Psi^{(+)}$ and $\Psi^{(-)}$, defined by (16.69), have the correct asymptotic forms, (16.42) and (16.46), respectively.

16.4 Use the Born approximation to calculate the differential cross section for scattering by the Yukawa potential, $V(r) = V_0\, e^{-\alpha r}/\alpha r$.

16.5 In Example 1 of Sec. 16.4 (spin–spin interaction), assume that the two particles are an electron and a proton, and add to H_0 the magnetic dipole interaction $-\mathbf{B}\cdot(\boldsymbol{\mu}_e + \boldsymbol{\mu}_p)$. Calculate the scattering cross sections in the Born Approximation, taking into account the fact that kinetic energy will not be conserved.

16.6 For Example 1 of Sec. 16.4 (without a magnetic field), assume that the phase shifts for the central potential $V_0(r)$ are known, and use the DWBA to calculate the additional scattering due to the spin–spin interaction $V_s(r)\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}$. Does skew scattering occur?

16.7 Show that Example 2 of Sec. 16.4 (spin–orbit interaction) can be solved "exactly" by introducing the total angular momentum eigenfunctions (7.104) as basis functions, and computing a new set of phase shifts that depend upon both the orbital angular momentum ℓ and the total angular momentum j. The solution will be as "exact" as the computation of the phase shifts. [Ref.: Goldberger and Watson (1964), Sec. 7.2.]